

Uniform approximation in various function systems.

by

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The aim of this paper is to establish approximation theorems of the Stone-Weierstrass-type <sup>in the</sup> system of all uniformly continuous real-valued mappings of a generalized uniform space (see [2]). As special cases we get theorems on uniformly continuous functions on (proper) uniform spaces (cf. theorem 4,1 of [5]), p-continuous functions, and continuous functions on completely regular topological spaces (cf. theorem 3,1 of [1]).

The development is in terms of uniform coverings. Although the setting thus differs from that of J.E. Fenstad (in [5], [6] and [7]) both in primary object of investigation and means of exposition, many of the ideas have their origin in his papers.

1. We start by recording some fundamental concepts from the theory of coverings of a set  $X$  (cf. [9], [10], [11]). The covering  $u$  is a refinement of a covering  $v$  - written  $u < v$  - if each  $U \in u$  is contained in some  $V \in v$ . If  $u$  is a covering and  $A$  a subset of  $X$ , the star  $St(A, u)$  is defined by the formula

$$St(A, u) = \bigcup \{U \in u : U \cap A \neq \emptyset\}.$$

The covering  $\{St(U, u) : U \in u\}$  is denoted  $u^*$ . If  $u^*$  is a refinement of  $v$ ,  $u$  is called a star-refinement of  $v$  - written  $u <^* v$ . A covering - or more generally a family -  $u$  is called star-finite if every  $U \in u$  meets only a finite number of sets in  $u$ .

If  $u$  and  $v$  are coverings, the product  $u \wedge v$  is defined by the formula

$$u \wedge v = \{U \wedge V : U \in u, V \in v\}.$$

A generalized uniformity on  $X$  may be defined as a collection  $\mathcal{U}$  of coverings which satisfies the following conditions

G 1. If  $u \in \mathcal{U}$ ,  $u < v$ , then  $v \in \mathcal{U}$ .

G 2. If  $u \in \mathcal{U}$ , there exists a  $v \in \mathcal{U}$  such that  $v <^* u$ .

G 3. If  $u$  and  $v$  are finite coverings in  $\mathcal{U}$ , then  $u \wedge v \in \mathcal{U}$ .

(cf. [2, p.251]).

A base of a generalized uniformity is defined in an obvious way.

A covering in  $\mathcal{U}$  is called a uniform covering (with respect to  $\mathcal{U}$ ). A generalized uniform space  $(X, \mathcal{U})$  is a set  $X$  equipped with a generalized uniformity  $\mathcal{U}$ . (The fundamental properties of generalized uniform spaces are developed in [2], then formulated in terms of entourages).

If  $\mathcal{U}$  in addition to G1 and G2 satisfies the condition

G 3'. If  $u$  and  $v$  are arbitrary coverings in  $\mathcal{U}$ , then  $u \wedge v \in \mathcal{U}$ ,

it is a (proper) uniformity.

A sequence  $\{u_n : n = 1, 2, \dots\}$  of coverings is called normal if  $u_{n+1} <^* u_n$  for all  $n$ . A normal sequence evidently constitute a base for a uniformity. If  $u_1 <^* u$ , the uniformity - as well as the sequence itself - are said to be associated with the covering  $u$ . Clearly to every  $u \in \mathcal{U}$  there exists an associated uniformity contained in  $\mathcal{U}$ .

If  $f$  is a mapping of a set  $X$  into a set  $Y$ , the inverse image  $f^{-1}(v)$  of a covering  $v$  of  $Y$  is defined to be the covering

$$f^{-1}(v) = \{f^{-1}(V) : V \in v\}.$$

A mapping  $f$  of a generalized uniform space  $(X, \mathcal{U})$  into a generalized uniform space  $(Y, \mathcal{V})$  is uniformly continuous if  $f^{-1}(v) \in \mathcal{U}$  for every  $v \in \mathcal{V}$ .

2. On the set of real numbers, the ordinary metric uniformity has a base consisting of all coverings  $u_\xi$ ,  $\xi > 0$ , where

$$u_\xi = \{ \langle (n-1)\xi, (n+1)\xi \rangle : n = 0, \pm 1, \pm 2, \dots \}$$

We shall let the letter  $R$  denote the lattice - ordered vector space of all real numbers, as well as this set equipped with the uniformity described.

The set of all uniformly continuous mappings of  $(X, \mathcal{U})$  into  $R$  shall be denoted  $U(X, \mathcal{U})$ . All the sets of real-valued mappings considered are thought of as subsystems of  $R^X$ , with the point-wise lattice - and vector space operations carried over to these systems, as far as they are defined. The constant mapping with value  $\rho$  shall simply be denoted  $\rho$ .

The system  $U(X, \mathcal{U})$  need constitute neither a group, nor a lattice. (Cf. the remark following theorem 3). Such anomalies are of course possible only if  $\mathcal{U}$  is not a proper uniformity. In any case,  $U(X, \mathcal{U})$  always contains all constant mappings and is closed under multiplication with real numbers. A subset of  $R^X$  which possesses these properties shall for shortness be called an m-system. In order to state some additional properties of  $U(X, \mathcal{U})$ , we give the following definitions.

A mapping  $f$  is said to separate the sets  $A$  and  $B$  if  $0 \leq f \leq 1$ ,  $f(A) = \{1\}$ ,  $f(B) = \{0\}$ , or vice versa. The family  $\{f_U : U \in u\}$  is said to separate the covering  $u$  if  $f_U(U) = \{1\}$  and  $f_U$  separates  $U$  and  $X - \text{St}(U, u)$  for every  $U \in u$ .

A family  $\{f_j : j \in I\}$  shall be called u-uniform if it is uniformly equi-continuous with respect to some uniformity  $\mathcal{U}_u \subset \mathcal{U}$  associated with  $u$ . That is: for every  $\xi > 0$  there shall exist a  $v \in \mathcal{U}_u$  which is a refinement of all  $f_j^{-1}(u_\xi)$ ,  $j \in I$ . A family which is u-uniform for some uniform covering  $u$  is simply called uniform.

We now state a lemma on suprema of families of real numbers.

Lemma 1. Let  $\{a_j : j \in I\}$  and  $\{b_j : j \in I\}$  be bounded families of real numbers. Then

$$\left| \sup\{a_j : j \in I\} - \sup\{b_j : j \in I\} \right| \leq \sup\{|a_j - b_j| : j \in I\}.$$

The proof is elementary and will be omitted (see p.ex. [5, p.437]).

Proposition 1. The m-system  $U(X, \mathcal{U})$  possesses the following properties:

- (1) The sum of any finite uniform family in  $U(X, \mathcal{U})$  belongs to  $U(X, \mathcal{U})$ . In particular,  $f + \varphi \in U(X, \mathcal{U})$  for  $\varphi \in R$ ,  $f \in U(X, \mathcal{U})$ .
- (2) The least upper bound of any uniform family in  $U(X, \mathcal{U})$  belongs to  $U(X, \mathcal{U})$ , in so far it exists as a finite mapping. In particular,  $\sup\{f, \varphi\} \in U(X, \mathcal{U})$  for  $\varphi \in R$ ,  $f \in U(X, \mathcal{U})$ .
- (3) For every uniform covering  $u$ ,  $U(X, \mathcal{U})$  contains a  $u$ -uniform family which separates  $u$ .

Proof. Statement (1) is obvious, while (2) immediately follows from the lemma cited. For the demonstration of (3) we make use of the following fact: Every normal sequence of coverings is determined by a pseudo-metric  $d$  in the sense that the collection  $\{d_\varepsilon : \varepsilon > 0\}$ , where  $d_\varepsilon$  consists of all sets of diameter less than  $\varepsilon$ , is a base for the uniformity determined by the sequence (cf. [4, p.15]). Now in particular let the normal sequence be associated with  $u$ , and all its elements contained in  $\mathcal{U}$ . If  $d_\varepsilon$  is a refinement of  $u$ , we define for every  $U \in u$ :

$$f_U(x) = \inf\left\{1, \frac{1}{\varepsilon} d(x, X - \text{St}(U, u))\right\}.$$

An easy argument shows that the family  $\{f_U : U \in u\}$  is contained in  $U(X, \mathcal{U})$ , is  $u$ -uniform and separates  $u$ .

3. We now embark on our primary task: the approximation theorems. To this end we need some more concepts.

For every real-valued mapping  $f$ , its cozero-set  $C(f)$  is defined by

$$C(f) = \{x \in X : f(x) \neq 0\}.$$

Further, we introduce a rather special kind of uniform coverings. For  $f \in U(X, \mathcal{U})$ ,  $\varepsilon > 0$ , we define

$$u(f, \varepsilon) = f^{-1}(u_\varepsilon).$$

(for the definition of  $u_\varepsilon$ , see section 2). A covering of this type will be called functionally determined. These coverings are not easy to give an intrinsic description, and we shall use them only provisionally. The main theorems will be formulated in terms of an important and easily described class of uniform coverings: Those which are star-finite and countable. (We remark that the word countable shall always mean "finite or countable infinite").

Proposition 2. Let  $(X, \mathcal{U})$  be a generalized uniform space, and  $S$  an  $m$ -system in  $U(X, \mathcal{U})$ . Assume that for every functionally determined covering  $u$  the following conditions are satisfied:

- (1) The sum and the supremum of any finite,  $u$ -uniform family in  $S$  belongs to  $S$ .
- (2) If  $\{f_j : j \in I\}$  is a countable,  $u$ -uniform family in  $S$  with the property that  $\{C(f_j) : j \in I\}$  is a star-finite covering of  $X$  refined by  $u$ , then  $\sup \{f_j : j \in I\} \in S$ .
- (3)  $S$  contains a  $u$ -uniform family which separates  $u$ .

In this case,  $S$  is uniformly dense in  $U(X, \mathcal{U})$ .

The proof is rather technical, and will be omitted. We remark that insignificant alterations of the proof may be made to the effect that the family in (3) need not be uniform if the families of (1) and (2) are not required to be uniform. The closedness conditions thus placed on  $S$  are very strong, and the corresponding theorem is in general not very useful. We shall, however, utilize it in section 6.

4. We now state the main theorems. We will call a family  $\{f_j : j \in I\}$  star-finite if  $\{C(f_j) : j \in I\}$  is star-finite, and we will say that it covers  $X$  if  $\{C(f_j) : j \in I\}$  is a covering of  $X$ . With this terminology an immediate specialization of prop. 2 yields:

Theorem 1 Let  $(X, \mathcal{U})$  be a generalized uniform space, and  $S$  an  $m$ -system in  $U(X, \mathcal{U})$ . Assume that for every countable, star-finite uniform covering  $u$  the following conditions are satisfied:

- (1)  $S$  is closed under formation of sums and suprema of finite,  $u$ -uniform families.
- (2)  $S$  is closed under formation of suprema of countable, star-finite  $u$ -uniform families which cover  $X$ .
- (3)  $S$  contains a  $u$ -uniform family which separates  $u$ .

In this case,  $S$  is uniformly dense in  $U(X, \mathcal{U})$ .

For convenience, we shall adopt the term l-space for a lattice-ordered vector space in  $R^X$  containing all constant mappings.

Theorem 2. Let  $(X, \mathcal{U})$  be a (proper) uniform space, and let  $S$  be an l-space in  $U(X, \mathcal{U})$  which satisfies the following conditions:

- (1)  $S$  is closed under formation of suprema of countable, star-finite uniform families which cover  $X$ .
- (2) For every countable, star-finite uniform covering  $u$ ,  $S$  contains a uniform family which separates  $u$ .

In this case,  $S$  is uniformly dense in  $U(X, \mathcal{U})$ .

Proof. If a family is  $v$ -uniform,  $v \in \mathcal{U}$ , it is also  $u \wedge v$ -uniform hence  $u$ -uniform. From this fact and prop. 2 the theorem immediately follows.

Remark. In [5] J.E. Fenstad has introduced conditions A(1) and A(2) which together secure uniform density of an l-space  $S$  in  $U(X, \mathcal{U})$ , in the case of proper uniformities. (Actually, the slightly more general situation where  $S$  is a lattice-ordered group containing all rational constants is considered). In order to formulate these conditions, we record some definitions. A strong  $u$ -cover is an ordered pair  $(V, u)$ , where  $u$  is a covering of  $X$ ,  $V$  an entourage of the uniformity, and there exists a number  $n$  such that for every  $A \in u$ ,  $V(A) \cap B \neq \emptyset$  for at most  $n$  sets  $B \in u$ . The cardinal number  $m(X, \mathcal{U})$  is defined by

$$m(X, \mathcal{U}) = \max \{ \aleph_0, \min \{ m; m > \text{card } u \text{ for all strong } u\text{-covers } (V, u) \} \}.$$

Now the conditions A(1) and A(2) are as follows:

A(1): To each entourage  $V$  of the uniformity, and each family  $\mathcal{F}$  of subsets, where  $\text{card } \mathcal{F} < m(X, \mathcal{U})$ , there exists a uniform family  $\{f_A : A \in \mathcal{F}\}$  in  $S$  such that  $f_A$  separates  $A$  and  $X - V(A)$ .

A(2):  $S$  is closed under formation of suprema of star-finite (locally finite in the terminology of [5]) uniform families of cardinality strictly less than  $m(X, \mathcal{U})$ .

If  $m(X, \mathcal{U}) > \aleph_0$ , A(1) and A(2) together with the reasoning following theorem 2 immediately secure conditions (2) and (3) of prop. 2. If on the other hand  $m(X, \mathcal{U}) = \aleph_0$ , i.e. all strong  $u$ -covers are finite, we see that all functionally determined coverings are finite. Every star-finite covering with a finite refinement is easily seen to be finite. Thus A(1) and A(2) entail (2) and (3) of prop. 2 also in this case. So we may conclude: If  $(X, \mathcal{U})$  is a (proper) uniform space and  $S$  an  $l$ -space in  $U(X, \mathcal{U})$  which satisfies A(1) and A(2), then  $S$  is uniformly dense in  $U(X, \mathcal{U})$  (cf. [5, p.438]).

5. We now consider a proximity space  $(X, \mathcal{P})$ . A covering  $u$  of  $X$  shall be called a proximity covering if

$$A \subseteq \text{St}(A, u) \quad \text{for all } A \subset X.$$

A proximity covering is normal if it possesses an associated normal sequence of proximity coverings. The set of all  $p$ -continuous mappings of  $(X, \mathcal{P})$  into the ordinary metric proximity space on the real numbers will be denoted  $P(X, \mathcal{P})$ . A family  $\{f_j : j \in I\}$  will be called  $u$ -uniform if it is uniformly equi-continuous with respect to some uniformity which is associated with  $u$  and contained in the collection of all normal proximity coverings. We may treat real-valued  $p$ -continuous mappings in the frame of the theory of uniformly continuous real-valued mappings, in view of the following facts:

The collection of all normal proximity coverings (with respect to  $\mathcal{P}$ ) constitutes a generalized uniformity  $\mathcal{U}_\mathcal{P}$ , and  $\mathcal{U}_\mathcal{P}$  is the finest generalized uniformity compatible with the proximity

(cf. [2, p.241]). A mapping between two proximity spaces is  $p$ -continuous if and only if it is uniformly continuous with respect to the corresponding finest generalized uniformities (cf. [2, p.246]). Every metric uniformity is the finest generalized uniformity compatible with its proximity ([2, p.243], [10, p.570]). Thus in particular  $P(X, \mathcal{P}) = U(X, \mathcal{U}_X)$ .

From theorem 1 and the facts recorded follows immediately:

Theorem 3. Let  $(X, \mathcal{P})$  be a proximity space, and  $S$  an  $m$ -system in  $P(X, \mathcal{P})$ . Assume that for every countable, star-finite, normal proximity covering  $u$  the following conditions are satisfied:

- (1)  $S$  is closed under formation of sums and suprema of finite,  $u$ -uniform families.
- (2)  $S$  is closed under formation of suprema of countable, star-finite  $u$ -uniform families which cover  $X$ .
- (3)  $S$  contains a  $u$ -uniform family which separates  $u$ .

In this case,  $S$  is uniformly dense in  $P(X, \mathcal{P})$ .

J.E. Fenstad has shown that  $P(X, \mathcal{P})$  - and a fortiori  $U(X, \mathcal{U})$  - need not be a group or a lattice. For an example see [7, p.135].

6. Now consider a completely regular topological space  $(X, \mathcal{T})$ . An open covering is normal if it has an associated normal sequence of open coverings. It is well known that the collection of all normal open coverings constitute a base of a uniformity  $\mathcal{U}_f$ , which is the finest uniformity compatible with the topology. Further  $U(X, \mathcal{U}_f) = C(X, \mathcal{T})$ , where  $C(X, \mathcal{T})$  denotes the set of real-valued continuous mappings of  $(X, \mathcal{T})$ . With an appropriate definition of uniform family we might thus formulate a theorem for  $C(X, \mathcal{T})$  analogous to theorem 2. We shall, however, only give a result on the lines indicated in the discussion following prop.2.

Two sets  $A$  and  $B$  are called normally separated if there exists a normal open covering  $u$  such that  $St(A, u) \cap B = \emptyset$ . This means exactly that there exists an  $f \in C(X, \mathcal{T})$  which separates  $A$  and  $B$ . (The term completely separated is also used for this concept, see p.ex. [3, p.252], [8, p.16]).



Theorem 4. Let  $(X, \mathcal{T})$  be a completely regular topological space. Assume that the l-space  $S$  in  $C(X, \mathcal{T})$  satisfies the following conditions:

- (1)  $S$  is closed under formation of suprema of countable, star-finite families which cover  $X$ .
- (2) Any two normally separated sets are separated by an element in  $S$ .

In this case,  $S$  is uniformly dense in  $C(X, \mathcal{T})$ .

Proof. Let  $u$  be a countable, star-finite, normal open covering. Condition (2) immediately secures the existence of a family in  $S$  which separates  $u$ . This together with the remarks following prop. 2 yields the desired result.

Remark. Anderson, in [3], has called a set  $S$  which satisfies condition (1) of theorem 4  $\sigma^*$ -complete. Further, he termed  $S$  normal if any two disjoint zero-sets (for continuous functions) are separated by an element in  $S$ . Now any two normally separated sets are contained in disjoint zero-sets (cf. [8, p.17]). Thus we may formulate as a corollary of theorem 4: Every normal  $\sigma^*$ -complete l-space in  $C(X, \mathcal{T})$  is uniformly dense in  $C(X, \mathcal{T})$ . Anderson further showed that the normality condition may be weakened to require only that for disjoint zero-sets  $Z_1$  and  $Z_2$  there exists an  $f \in S$  such that  $f^{-1}(\{0\}) \supset Z_1$ ,  $f^{-1}(\langle 0, \infty \rangle) \supset Z_2$ .

We close by deducing a version of the ordinary Stone-Weierstrass theorem.

Theorem 5. Let  $(X, \mathcal{T})$  be a compact space, and  $S$  an l-space in  $C(X, \mathcal{T})$ . If every two distinct points are separated by an element in  $S$ , then  $S$  is uniformly dense in  $C(X, \mathcal{T})$  (cf. [5, p.242]).

Proof. In this case every open covering has a finite refinement, hence every star-finite open covering is finite. Thus condition (1) of theorem 5 is certainly satisfied. If  $A$  and  $B$  are normally separated,  $\bar{A}$  and  $\bar{B}$  are disjoint. A standard argument shows that if every pair of points is separated by an element of  $S$  then also any two disjoint compact sets are separated by an element in  $S$  (see p.ex. [5, p.242]). Hence the desired conclusion follows.

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